CS 518: Problem Set 3

Section: MW 12-1:30 pm

Total: 100pts Due: 11/16/2015

Problem 1 Simon's Algorithm: 28 pts

Suppose we run Simon's algorithm on the following input x (with $N = 8$ and hence $n = 3$): We have
$x_{000} = x_{111} = 000 \qquad x_{001} = x_{110} = 001$
$x_{010} = x_{101} = 010 \qquad x_{011} = x_{100} = 011$
Note that x is 2-to1 and $x_i = x_{i\oplus 111}$ for all $i \in \{0,1\}^3$, so $s = 111$. (a) Give the starting state of Simon's algorithm.
(b) Give the state after the first Hadamard transforms on the first 3 qubits.
(c) Give the state after applying the oracle.
(d) Give the state after measuring the second register (the measurement gave $|001\rangle$.
(e) Use $H^{\otimes n}|i\rangle = \frac{1}{\sqrt{2}} \sum_{j\in\{0,1\}^n} (-1)^{i\cdot j}|j\rangle$, give the state after the final Hadamard.
(f) We does measurement of the first 3 qubits of the final state give the information about s?
(g) Suppose the first run the the algorithm gives $j = 011$ and a second run gives $j = 101$. Show that, assuming $s \neq 000$, those two runs of the algorithm already determine s.

Problem 2 Fourier Transform: 30 pts

(a) For $\omega = e^{2\pi i/3}$ and $F_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$, calculate $F_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $F_3 \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$.
(b) Let the Fourier transform defined as what we described in class, ie. $\omega_= e^{2\pi i/N}$ and entry at location (i, j) is $e^{2\pi ijk/N}$ where $0 \leq j, k < N$. Let $|C_k\rangle$ be the k_{th} column of F_N. Please show that

$$\langle C_k' | C_k \rangle = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}$$

(c) Prove the identity in equation 7.1.18 in the textbook.

Problem 3 Euclidean distance: 12 pts

The *Euclidean distance* between two states $|\phi\rangle = \sum_i \alpha_i|i\rangle$ and $|\psi\rangle = \sum_i \beta_i|i\rangle$ as $|||\phi\rangle - |\psi\rangle|| = \sqrt{\sum_i |\alpha_i - \beta_i|^2}$ Assume the states are unit vectors with real amplitudes. Suppose the distance is small: $|||\phi\rangle - |\psi\rangle|| = \epsilon$. Show that the probabilities resulting from a measurement on the two states are also close: $\sum_i |\alpha_i^2 - \beta_i^2| \leq 2\epsilon$. (Hint use *Cauchy-Schwarz* inequality)

Problem 4 Analysis Technique Proof: 10 pts

In quantum counting or the hard case analysis of Shor's algorithm, the following analysis technique is commonly used: $|1 - e^{i\theta}| = 2|\sin(\frac{\theta}{2})|$ Please prove this equality.

Problem 5 Root of Unity: 10 pts

Prove that if a operator U satisfies $U^r = I$, then the eigenvalues of U must be rth roots of 1.

Problem 6 Gate Approximation: 10 pts

As mentioned in class that the implementation of QFT inverse will be difficult if the precision requirement is high for the control rotation gates. Based on the Solovay-Kitaev's decomposition theorem, there is always a way to approximate a single qubit gate with error at most ϵ using $O(log^c(1/\epsilon))$ gates from the universal gate where the optimal c is some number slightly less than 2. Please describe (sketch) the proof of this theorem.