MAT 115: Exam 3

Instructions:

You have 70 minutes for this exam. There are 7+1 problems. Calculators are allowed. The total score is 85+15 points with perfect score of 100 points. Please work on those in which you have confidence. If you finish the exam early, please just leave your exam on the desk and I will collect it. I leave plenty of space on each page for your computation. One extra blank sheet is also attached at the end of this exam if you need more computation space. If your answer is incorrect but you show the computation process, then partial credits will be given. Please use your time and space wisely.

First Name:

Last Name:

Score: **/100**

Bonus: /10

Total: /110

Problem 1: Combinatorics Summation (10 pts)

Prove that the number of subsets of a set S, including the empty set and S itself, is $2^{|S|}$.

Problem 2 Functions (10 Points)

Let A and B be finite sets and f: A → B. Prove the following claims. Some are practically restatements of the definitions; some require a few steps.

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

(c) If f is a bijection, then $|A| = |B|$.

(d) If $|A| = |B|$, then f is an injection if and only if it is a surjection.

(e) If $|A| = |B|$, then f is a bijection if and only if it is an injection or it is a surjection.

Problem 3: Lists Permutations (15 pts)

We want to know how many ways 3 boys and 4 girls can sit in a row.

(a) How many ways can this be done if there are no restrictions?

(b) How many ways can this be done if the boys sit together and the girls sit together?

(c) How many ways can this be done if the boys and girls must alternate?

Problem 10: Basic Counting and Listing (10 pts)

Suppose 3 different balls are placed into 3 labeled boxes at random.
(a) What is the probability that no box is empty?

(b) What is the probability that exactly one box is empty?

(c) What is the probability that at least one box is empty?

(d) Repeat (a)–(c) if there are 5 balls and 4 boxes.

Problem 5 Lists with Repetitions (10 points)

How many different three-digit positive integers are there? (No leading zeroes are allowed.) How many positive integers with at most three digits? What are the answers when “three” is replaced by “n?”

Problem 6: Permutations (15 pts)

This exercise deals with powers of permutations. All our permutations will be written in cycle form.

(a) Compute $(1, 2, 3)^{200}$.

(b) Compute $((1, 3)\ (2, 5, 4))^{200}$.

(c) Show that for every permutation f of 5, we have f^{60} is the identity permutation. What is f^{61}

Problem 7: Permutations (15 pts)

This exercise lets you check your understanding of cycle form. A permutation is given in one-line, two-line or cycle form. Convert it to the other two forms. Give its inverse in all three forms.

(a) (1,4,7,8) (2,3) (7) (8).

(b) (4,2,3,2,1), which is in cycle form.

(c) (5,4,3,2,1), which is in one-line form.

Bonus: Probability and Basic Counting (15 pts)

For n > 0, prove the following formulas for S(n, k):

$$S(n,n) = 1, \qquad S(n,n-1) = \binom{n}{2}, \qquad S(n,1) = 1, \qquad S(n,2) = (2^n-2)/2 = 2^{n-1}-1.$$

BASIC ALGEBRAIC RULES:

Theorem 2 (Algebraic rules for Boolean functions) *Each rule states that two different-looking Boolean functions are equal. That is, they look different but have the same table.*

Associative Rules:	$(p \wedge q) \wedge r = p \wedge (q \wedge r)$	$(p \vee q) \vee r = p \vee (q \vee r)$
Distributive Rules:	$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
Idempotent Rules:	$p \wedge p = p$	$p \vee p = p$
Double Negation:	$\sim\sim p = p$	
DeMorgan's Rules:	$\sim(p \wedge q) = \sim p \vee \sim q$	$\sim(p \vee q) = \sim p \wedge \sim q$
Commutative Rules:	$p \wedge q = q \wedge p$	$p \vee q = q \vee p$
Absorption Rules:	$p \vee (p \wedge q) = p$	$p \wedge (p \vee q) = p$
Bound Rules:	$p \wedge 0 = 0 \qquad p \wedge 1 = p$	$p \vee 1 = 1 \qquad p \vee 0 = p$
Negation Rules:	$p \wedge (\sim p) = 0$	$p \vee (\sim p) = 1$

SCRATCH WORK