MAT 115: Problem Set 4

Due: 12/02/2019

Instructions:

I leave plenty of space on each page for your computation. If you need more sheet, please attach your work right behind the corresponding problem. If your answer is incorrect but you show the computation process, then partial credits will be given. Please staple your solution and use the space wisely.

First Name:

Last Name:

Score: /130

Problem 1: Combinatorics Summation(10 pts)

Prove that the number of subsets of a set S, including the empty set and S itself, is $2^{|S|}$.

Problem 2: Lists with Repetitions (10 pts)

How many different three-digit positive integers are there? (No leading zeroes are allowed.) How many positive integers with at most three digits? What are the answers when “three” is replaced by “n?”

Problem 3: Lists Without Repetition (15 pts)

We want to know how many ways 3 boys and 4 girls can sit in a row.

(a) How many ways can this be done if there are no restrictions?

(b) How many ways can this be done if the boys sit together and the girls sit together?

(c) How many ways can this be done if the boys and girls must alternate?

Problem 4: Lists Without Repetition (10 pts)

How many ways are there to form a list of two letters from the set of letters in the word COMBINATORICS if the letters cannot be used more often than they appear in COMBINATORICS? three letters?

Problem 5: Functions (10 pts)

Let A and B be finite sets and f: A → B. Prove the following claims. Some are practically restatements of the definitions; some require a few steps.

(a) If f is an injection, then $|A| \leq |B|$.

(b) If f is a surjection, then $|A| \geq |B|$.

Problem 6: Permutations (15 pts)

This exercise deals with powers of permutations. All our permutations will be written in cycle form.

(a) Compute $(1, 2, 3)^{300}$.

(b) Compute $((1, 3)\ (2, 5, 4))^{300}$.

(c) Show that for every permutation f of 5, we have f^{60} is the identity permutation. What is f^{61}

Problem 7: Permutations (15 pts)

This exercise lets you check your understanding of cycle form. A permutation is given in one-line, two-line or cycle form. Convert it to the other two forms. Give its inverse in all three forms.

(a) (1,5,7,8) (2,3) (4) (6).

(b) (5,4,3,2,1), which is in cycle form.

(c) (5,4,3,2,1), which is in one-line form.

Problem 8: Permutations (15 pts)

(a) In how many ways can eight books be arranged along a shelf?

(b) At Scrabble the letters QWYPKGDZXBM are left in the bag. In how many ways can you draw out four of them?

(c) You are given 12 points on a plane, no three of them being in a straight line. How many triangles can be drawn using the points as vertices?

Problem 9: Probability and Basic Counting (10 pts)

Prove $\binom{n}{k} = \binom{n}{n-k}$ and $\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$.

Problem 10: Basic Counting and Listing (10 pts)

Suppose 4 different balls are placed into 4 labeled boxes at random. (This can be done in 4^4 ways.)

(a) What is the probability that no box is empty?

(b) What is the probability that exactly one box is empty?

(c) What is the probability that at least one box is empty?

(d) Repeat (a)–(c) if there are 5 balls and 4 boxes.

Problem 11: Basic Counting and Listing (10 pts)

An urn contains ten labeled balls, labels 1, 2, . . . , 10.

(a) Two balls are drawn together. What is the sample space? What is the probability that the sum of the labels on the balls is odd?

(b) Two balls are drawn one after the other without replacement. What is the sample space? What is the probability that the sum is odd?